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A general model for simulating the effect of the intercommunication between various elements of a structure on the extension of a crack

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The characteristics of the intercommunication between the various elements of a structure have a major effect on the structure's resistance to failure by the extension of a crack. Two extremes are: (a) where the intercommunication is very strong when the structure is particularly vulnerable to crack extension, and (b) where the elements are almost totally isolated when the structure is highly resistant to crack extension. This paper presents a very simple and general model that quantifies the effect of the intercommunication on a structure's crack extension resistance, and unifies the various types of intercommunication within a single model analysis.

1. Introduction

The ease with which crack extension occurs in a structure depends on the extent to which the applied strain is focused at the crack tip. This focusing depends in turn on the characteristics of the intercommunication between the various elements that comprise the structure. In a recent review Gordon [1] has identified four main types of intercommunication;

1. The intercommunication is very strong, as in a brittle crystalline material, where the strain is readily focused because the material's atomic bonds are highly resistant to shear; the structure is then particularly vulnerable to crack extension.
2. The intercommunication is very weak and the structural elements are almost totally isolated, examples being rope, cloth, the cables of modern suspension bridges and natural tendons; the structure is then highly resistant to crack extension.
3. Systems containing weak interfaces where the adhesion between the structural elements breaks down fairly readily, examples being timber, teeth and some artificial composite materials; the intercommunication is then essentially weak and the structure is highly resistant to crack extension.
4. Systems where the intercommunication is weak at low shear strains followed by a rapid increase at a critical strain level so that the shear resistance is very high near a crack tip; examples are many human and animal membranes, which are highly resistant to crack extension.

This paper presents a very simple and general model that quantifies the effect of the intercommunication on the resistance of a structure to crack extension, and unifies the various types of intercommunication within a single model analysis.

2. General theoretical analysis

The two-dimensional model simulating the effect of the intercommunication between the various elements of a structure on the extension of a crack is illustrated in Fig. 1. There are three horizontal linear elastic elements each of length $2h$ and of tensile modulus M. To simulate a displacement applied to the system, the terminal points of each element are subjected to applied displacements $\pm D_*$. The outer elements are assumed to have fractured at their mid-points and the crack extension process is simulated by the fracturing of the central element at its mid-point, it is assumed that this requires the attainment of a critical fracture strain ε_F. It must be emphasized that this model is highly idealized in the manner in which it simulates the crack extension process since only three elements are involved in the cracking process; however, it is expected that similar results will ensue from the analysis of models involving more linear elements, with the number broken being representative of the crack size. The shear intercommunication between the elements is simulated by a restraining force between adjacent elements that is a function of the relative displacement of equivalent points in adjacent elements.

With x being measured from the mid-points of the elements, the displacements $u(x)$ of the outer elements and the displacement $w(x)$ of the central element are odd functions of x. If $T_u(x)$ is the tension within an outer element, $T_w(x)$ is the tension within the central element, and $f(\phi) \equiv f(u - w)$ is the restraining force per unit length provided by the shear intercommunication between the elements, equilibrium of the elements provides the equations

$$\frac{dT_u}{dx} - f(u - w) = 0 \qquad (1)$$

$$\frac{dT_w}{dx} + 2f(u - w) = 0 \qquad (2)$$

with

$$T_u = M\frac{du}{dx} \qquad (3)$$

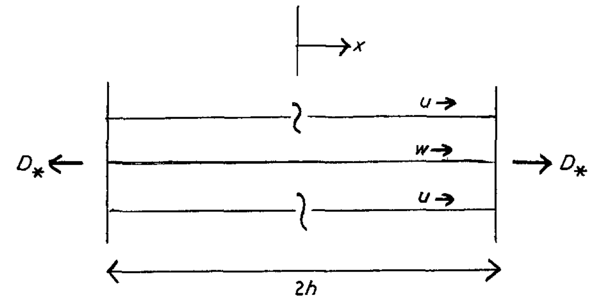

Figure 1 The simple model simulating the effect of the shear intercommunication between the various elements of a structure on the extension of a crack. $2h$ is the length of each element and b is the spacing between the elements.

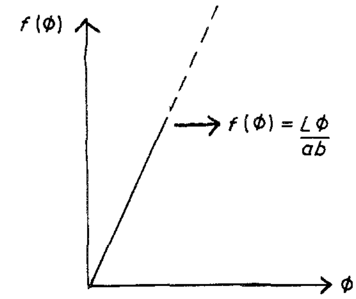

Figure 2 Linear intercommunication law.

and

$$T_w = M\frac{dw}{dx} \tag{4}$$

whereupon Equations 1 and 2 become respectively

$$\frac{d^2u}{dx^2} - \frac{f(u-w)}{M} = 0 \tag{5}$$

and

$$\frac{d^2w}{dx^2} + \frac{2f(u-w)}{M} = 0 \tag{6}$$

By virtue of the system's symmetry it is necessary to consider only one half of the system (i.e. $x > 0$) for which the boundary conditions are

$$u, w = D_* \quad \text{when } x = h \tag{7}$$

$$\frac{du}{dx}, w = 0, \quad \text{when } x = 0 \tag{8}$$

the condition $du/dx = 0$ when $x = 0$ reflecting the fact that the outer elements are fractured at their mid-points.

It follows from Equations 5 and 6 that

$$\frac{d^2}{dx^2}[2u + w] = 0 \tag{9}$$

and

$$\frac{d^2\phi}{dx^2} - \frac{3f(\phi)}{M} = 0 \tag{10}$$

with $\phi \equiv u - w$. Integration of Equation 9 gives

$$2u + w = A + Bx \tag{11}$$

where A and B are constants that are yet to be determined, while integration of Equation 10 gives

$$\frac{d\phi}{dx} = -\left(C + \frac{6}{M}\int_0^\phi f(\psi)\, d\psi\right)^{1/2} \tag{12}$$

where C is a constant that is yet to be determined. Integration of Equation 12 gives

$$(h - x) = \int_0^\phi \frac{dx}{\left(C + \frac{6}{M}\int_0^x f(\psi)\, d\psi\right)^{1/2}} \tag{13}$$

after noting (from the Boundary Condition 7) that $\phi = 0$ when $x = h$. Supposing that: (a) the value of ϕ at $x = 0$ is ϕ_0, (b) the local strain at the mid-point of the central element is $\varepsilon_L = (dw/dx)_{x=0}$, and (c) the macroscopic fracture strain D_*/h is equal to ε_M, then it follows from the Boundary Conditions 7 and 8, together with Equations 11, 12 and 13 that

$$h = \int_0^{\phi_0} \frac{dx}{\left(\varepsilon_L^2 - \frac{6}{M}\int_x^{\phi_0} f(\psi)\, d\psi\right)^{1/2}} \tag{14}$$

and

$$\frac{2\phi_0}{h} + \varepsilon_L = 3\varepsilon_M \tag{15}$$

Equations 14 and 15 allow the effect of shear intercommunication on the crack extension process to be assessed, in that they enable the influence of the form of $f(\psi)$ on the magnitude of the local strain ε_L to be assessed. If there is no shear intercommunication between the elements and $f(\psi) = 0$, Relations 14 and 15 immediately show that $\varepsilon_L/\varepsilon_M = 1$ and there is no focusing of the applied strain. At the other extreme where the intercommunication is so strong that the elements are rigidly connected, $\varepsilon_L/\varepsilon_M = 3$ and the strain is focused by a factor of three. Relations 14 and 15 will now be used to assess the effect of various types of intercommunication on the extent to which the strain is focused.

3. Linear intercommunication

If the intercommunication between the structural elements is linear, $f(\phi)$ can be written as $L\phi/ab$ where b is the distance between the elements and a is a materials related length parameter associated with each element, i.e.

$$f(\phi) = L\phi/ab \tag{16}$$

as shown in Fig. 2. In this case relations (14), (15) and (16) give

$$\varepsilon_L/\varepsilon_M = 3\Big/\left\{\frac{2\tanh[3L/Mab]^{1/2}h}{[(3L/Mab)^{1/2}h]} + 1\right\} \tag{17}$$

Relation 17 shows that if there is no intercommunication between the elements, i.e. $L = 0$, then $\varepsilon_L/\varepsilon_M = 1$, while if the elements are rigidly connected, i.e. $L = \infty$, then $\varepsilon_L/\varepsilon_M = 3$; these conclusions are in accord with the conclusions reached in the preceding section. The ratio $\varepsilon_L/\varepsilon_M$ for different values of the parameter $h(3L/Mab)^{1/2}$ is shown in Table I; these results clearly show how the strength of the intercommunication, as manifested by the parameter L,

TABLE I The ratio $\varepsilon_L/\varepsilon_M$ for different values of the parameter $h(3L/Mab)^{1/2}$ for a linear intercommunication law

$h(3L/Mab)^{1/2}$	$\varepsilon_L/\varepsilon_M$
0	1
1	1.19
2	1.53
5	2.14
10	2.50
∞	3

affects the crack extension process, as manifested by the magnitude of the ratio $\varepsilon_L/\varepsilon_M$. Relation 17 also shows that $\varepsilon_L/\varepsilon_M = 3$ as $h \to \infty$, irrespective of the strength of the intercommunication, i.e. of the value of L.

4. Intercommunication involving weak interfaces

Intercommunication involving weak interfaces can be described by the force law

$$\left.\begin{array}{lll} f(\phi) = L\phi/ab & & 0 < \phi < \phi_* \\ f(\phi) = 0 & & \phi > \phi_* \end{array}\right\} \tag{18}$$

as shown in Fig. 3. In this case, Relations 15 and 17 show that the condition for weak interfaces to exist is

$$\phi_* < \frac{\varepsilon_F \tanh\left(\dfrac{3L}{Mab}\right)^{1/2} h}{\left(\dfrac{3L}{Mab}\right)^{1/2}} \tag{19}$$

where ε_F is the critical value of the local strain ε_L at which an element fractures. It follows from Relations 14, 15 and 18 that, when there are weak interfaces (i.e. Condition 19 is satisfied), the macroscopic fracture strain is given by the expression

$$\varepsilon_M = \frac{2\phi_*}{3h} + \varepsilon_L\left[1 - \frac{2}{3h}\left(\frac{Mab}{3L}\right)^{1/2} \times \tanh^{-1}\left(\frac{3L}{Mab}\right)^{1/2}\frac{\phi_*}{\varepsilon_L}\right] \tag{20}$$

This expression can be written in the form

$$\frac{\varepsilon_M}{\varepsilon_F} = 1 - \frac{2}{3h}\left(\frac{Mab}{3L}\right)^{1/2}\left\{\tanh^{-1}\left[k\tanh\left(\frac{3L}{Mab}\right)^{1/2} h\right] - k\tanh\left(\frac{3L}{Mab}\right)^{1/2} h\right\} \tag{21}$$

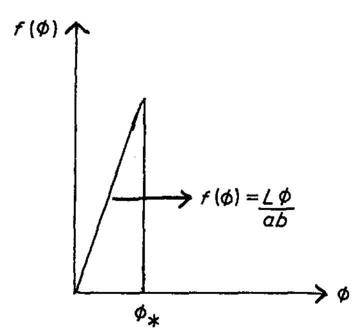

Figure 3 Intercommunication involving weak interfaces. The intercommunication is linear for $\phi < \phi_*$, while there is no intercommunication for $\phi > \phi_*$.

where the parameter k is given by the expression

$$k = \phi_* \Bigg/ \left[\frac{\varepsilon_F \tanh\left(\dfrac{3L}{Mab}\right)^{1/2} h}{\left(\dfrac{3L}{Mab}\right)^{1/2}}\right] \tag{22}$$

and varies between zero, when there is no intercommunication between the structural elements, and unity, when there are no weak intefaces. At these two extremes, the macroscopic fracture strain has the values

$$\varepsilon_M = \varepsilon_F \text{ (no intercommunication: } k = 0) \tag{23}$$

$$\varepsilon_M = \varepsilon_F\left\{\frac{1}{3} + \frac{2}{3}\left[\frac{\tanh\left(\dfrac{3L}{Mab}\right)^{1/2} h}{\left(\dfrac{3L}{Mab}\right)^{1/2} h}\right]\right\}$$

$$\text{(No weak interfaces: } k = 1) \tag{24}$$

As an illustration of the fact that weak interfaces make crack extension more difficult, $\varepsilon_M/\varepsilon_F = 1$ when $k = 0$ and $\varepsilon_M/\varepsilon_F = 0.84$ when $k = 1$ for the particular case where $h(3L/Mab)^{1/2}$ is equal to unity. The corresponding values for $h(3L/Mab)^{1/2} = 5$ are $\varepsilon_M/\varepsilon_F = 1$ when $k = 0$ and $\varepsilon_M/\varepsilon_F = 0.47$ when $k = 1$.

5. Weak intercommunication at low strains

The case where the intercommunication is weak at low strains can be considered by reference to the idealized force law

$$\left.\begin{array}{lll} f(\phi) = 0 & & \phi < \phi < \phi_* \\ f(\phi) = \dfrac{L(\phi - \phi_*)}{ab} & & \phi > \phi_* \end{array}\right\} \tag{25}$$

as shown in Fig. 4. Clearly ϕ_* must be less than $h\varepsilon_F$ in order for the intercommunication to have an effect on the fracture process; if this condition is satisfied, it follows from Relations 14, 15 and 25 that the macroscopic fracture strain is given by the expression

$$h = \frac{\phi_*}{\varepsilon_F\left[1 - \dfrac{3L(\phi_0 - \phi_*)^2}{Mab\,\varepsilon_F^2}\right]} + \left(\frac{Mab}{3L}\right)^{1/2} \times \tanh^{-1}\left[\left(\frac{3L}{Mab}\right)^{1/2}\frac{(\phi_0 - \phi_*)}{\varepsilon_F}\right] \tag{26}$$

with

$$\frac{2\phi_0}{h} + \varepsilon_F = 3\varepsilon_M \tag{27}$$

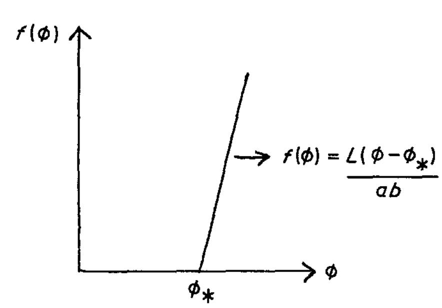

Figure 4 Force law for which there is no intercommunication if $\phi < \phi_*$ and a linear intercommunication if $\phi > \phi_*$.

TABLE II The ratio $\varepsilon_M/\varepsilon_F$ for different values of $\mu \equiv \phi_*/h\varepsilon_F$ for the special case where $h(3L/Mab)^{1/2}$ is equal to unity

$\mu = \dfrac{\phi_*}{h\varepsilon_F}$	$\dfrac{\varepsilon_M}{\varepsilon_F}$
0	0.841
0.095	0.863
0.390	0.927
0.659	0.972
0.895	0.997
1.000	1.000

Thus writing $(\phi_*/h\varepsilon_F) = \mu$ and noting that μ varies between zero and unity, Expressions 26 and 27 can be written in the forms

$$1 = \frac{\mu}{\left[1 - \dfrac{3Lh^2}{Mab}\dfrac{(\phi_0 - \phi_*)^2}{h^2\varepsilon_F^2}\right]^{1/2}} + \frac{1}{h}\left(\frac{Mab}{3L}\right)^{1/2} \times \tanh^{-1}\left[h\left(\frac{3L}{Mab}\right)^{1/2}\frac{(\phi_0 - \phi_*)}{h\varepsilon_F}\right] \quad (28)$$

and

$$\gamma = \frac{(\phi_0 - \phi_*)}{h\varepsilon_F} = \frac{3\varepsilon_M}{2\varepsilon_F} - \frac{(1 + 2\mu)}{2} \quad (29)$$

To examine the effect of the force law on the crack extension process, consider the special case where $h(3L/Mab)^{1/2}$ is equal to unity when Expressions 28 and 29 simplify to

$$\mu = (1 - \gamma^2)^{1/2}(1 - \tanh^{-1}\gamma) \quad (30)$$

and

$$\frac{\varepsilon_M}{\varepsilon_F} = \tfrac{2}{3}(\gamma + \mu + 1/2) \quad (31)$$

Relations 30 and 31 allow $\varepsilon_M/\varepsilon_F$ to be obtained for different values of μ, and the results are shown in Table II. These results illustrate the beneficial effect, with regards to the prevention of crack extension, of the shear resistance being low at small strains, in that the table shows how $\varepsilon_M/\varepsilon_F$ increases with ϕ_*.

6. Discussion

This paper has presented a very simple and general model that provides a quantitative assessment of the effects of the shear intercommunication between the various elements of a structure on the structure's resistance to crack extension. Various types of intercommunication have been unified within a single model description and have been treated as special cases, e.g. (a) where the intercommunication is linear, (b) where the intercommunication breaks down at a critical shear strain, and (c) where the intercommunication is weak at low shear strains and increases at a critical shear strain. The results of the analyses for these special cases underscore Gordon's view [1] that the characteristics of the shear intercommunication have a crucial effect on a structure's resistance to crack extension.

The author has already considered the case where the shear intercommunication is linear (see force law in Fig. 2) using a model [2] in which the shear intercommunication is provided by discrete linkages, and obtained results that are analogous and similar to those obtained in Section 3. He has also used a Mode II discrete linkage model [3] to show that the resistance to crack extension is large if the intercommunication is weak at low shear strains and increases markedly as the strain increases above a critical value. However, the author believes that it is more appealing to be able to consider a wide range of intercommunication characteristics within the umbrella of a single model. This has been the objective of the present paper, and to obtain simple relationships it is necessary to have a continuum rather than a discrete linkage description of the shear intercommunication.

It must be emphasized that the model is highly idealized in the way in which the crack extension process is simulated. There are only three elements in the model and it would obviously be more realistic to have a larger number of elements with the number of fractured elements being representative of the crack size. Nevertheless the model does highlight the salient features of the effects of shear intercommunication on crack extension, and does so in a manner that is easily understood. Because of the model's generality, there is clearly appreciable scope for further extension of its usage. For example it can in principle be used to study the competition between slip at a crack tip and crack extension, by using an appropriate shear intercommunication law to describe the slip process. It should therefore be possible to develop criteria for defining whether crack extension occurs with or without plastic deformation, thus complementing the results from earlier analyses [4, 5].

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